

Software Goes to School

Teaching for Understanding with New Technologies

Edited by

David N. Perkins

Judah L. Schwartz

Mary Maxwell West

Martha Stone Wiske

New York Oxford
OXFORD UNIVERSITY PRESS
1995

8

Creating Cybernetic and Psychological Ramps from the Concrete to the Abstract: Examples from Multiplicative Structures

JAMES J. KAPUT

This chapter will reflect on several years' work of the ETC Multiplicative Structures/Word Problems Project from a representational perspective. The plan of the chapter is (1) to recount enough history to situate the reflection, (2) to provide a touchstone illustrative example based on our work, (3) to offer a theoretical context in which to interpret the example, (4) to revisit the work in more detail using the framework, and (5) to reflect on the entire project in ways that may help others see a more general approach to using technology to attack other mathematical targets of difficulty. Details regarding results of empirical work and software development are available in a variety of publications listed in the References and mentioned, as appropriate, throughout the chapter.

Capsule Background on the Project

The Mathematics of Quantity Versus the Arithmetic of Pure Number

The history of our effort helps reveal the source of the Project's dual title. The group of mathematics teachers and researchers who were among those who defined ETC's targets of difficulty decided at the outset that student difficulty with word problems was to be their target. Given all the research that had been done in that area, it was decided to focus on problems involving multiplication, division, rates, ratio, and proportion, a web of ideas now known as the conceptual field of multiplicative structures

(Vergnaud, 1983, 1988). Further studies revealed the center of this web to be the complex and subtle idea of "intensive quantity," especially as used to model multiplicative relationships.

The reader can think of *intensive quantities* as those used to express comparative relationships involving the "per" word, such as 6 candies per child, 30 miles per hour, 3 feet per yard, 2 dollars per 3 hot dogs, 3 grams per 2 liters, and so on. These include ratios, densities, and rates as well as scale conversion factors. Intensive quantities are usually made up of *extensive quantities*, which measure the "number of" or "measure of." Examples include 6 hot dogs, 30 miles, 4½ hours, 1.7 liters, and so on. Our working mathematical theory of quantity is well expressed by Schwartz (1988). Of course, the psychological aspects of quantity are quite another matter (Kaput and West, 1993).

The group's research agenda began with initially identifying behavioral difficulties, mainly trouble with word problems involving intensive quantities, and turned to ascertaining the important underlying cognitive structures associated with both good as well as poor performance. We then focused on determining how students could be helped to build cognitive structures that would support this web of ideas and what role technology might play in this process.

Our Guiding Assumptions

Three early assumptions have marked our work. The first two are curricular, the third a software design assumption. I will list and discuss these briefly.

The early and middle school mathematical experience should center on the mathematics of quantity rather than the arithmetic of pure number. Simply put, we have assumed throughout that children's mathematics should be *about* something—all numbers refer to elements or relationships in the child's experience, so that we never ask a child to compute or otherwise reason with pure numbers (unless there is some purpose-giving context for the activity); for example, we would not ask the question "What number do you need to multiply 7 by to get 42?" We might, however, put the child in the position of needing to know how many candies would each of 7 classmates get if 42 candies were distributed evenly among them. Or, in a different situation, how many children could get 7 candies apiece if there were 42 candies to be distributed evenly among them. The reader will notice that the formal pure-number calculation normally used to solve both of these two very different problems is the same for each, namely the division of 42 by 7, $42/7$. Of course, the reasoning that leads to the division procedure, and even the execution of the division procedure itself, depends on the conceptualization of the respective situations. Furthermore, the critical difference between the models for the situations is in the role of the associated intensive quantities. The first starts with the two (extensive) quantities 42 candies and 7 classmates and generates the inten-

sive quantity 42 candies per 7 classmates, which reduces to the intensive quantity 6 candies per classmate. The second requires a division of the (extensive) quantity 42 candies by the intensive quantity 7 candies per child, with a resulting extensive quantity answer of 6 children. The arithmetic of pure number largely disacknowledges the difference between these two situations and the conceptualizations needed to model them. It is thus little surprise, but enormously important, that students whose experience has been limited to the arithmetic of number have difficulty with such modeling acts.

The multiplicative structures curriculum should be longitudinally coherent. We assert first that multiplication, division, and rate and ratio should fit together both mathematically, as relating to the several ways that intensive quantities are involved, and conceptually, in terms of the cognitive growth of interconnections among their associated conceptual structures. We assert second that the ideas of rate, ratio, and proportion (an equality-comparison of ratios or rates, i.e., of intensive quantities) should be treated as special cases of linear quantitative relationships, eventually expressible as linear functions. This means that we represent and deal with these ideas in ways that extend smoothly to the mathematics of more general functional relationships, which in turn means that we use tables of data, coordinate graphs, and algebraic equations to represent these ideas. Our curricular assumptions are described in more detail in our 1985 Technical Report (Kaput, 1985) and in Kaput and West (1993).

The goal of computer software should be to help students extend what they know from familiar, concrete contexts to less familiar, abstract contexts by cybernetically linking more familiar representations to less familiar ones.

Thus, our software-based learning environments were not designed to facilitate computation directly (as calculators do), or provide direct "right/wrong" feedback, as classic computer-assisted Instruction did. Instead, it engages the students in a series of representation systems, beginning at the concrete level and "ramping upward" in abstractness to more abstract representation systems. Furthermore, these representations are systematically linked, so that the student typically is in a situation where the results of an action or choice in one representation can be viewed in another. In this way, we employ the student's sense-making power to render judgment on the adequacy or appropriateness of actions or choices that they themselves make. The computer reports and the student evaluates, instead of the reverse, which has often been the case historically. Thus, the computer is a structured medium for reasoning. The concrete representations involve manipulating objects on the computer screen—icons representing the entities in the situation being modeled—to enact multiplication, division, and ratio reasoning. We will refer to these as "Icon Calculation Environments" (ICE). These are intended to capitalize on the rich physical experience students have had with objects, especially with grouping and counting them and with counting the groups they might be arranged into (Kaput, in press a).

A Touchstone Example

I will now offer an example of how a concrete representation used in one of the ICE programs may be used to solve a standard missing-values proportion problem by using only grouping and counting actions. I will follow with an illustration of how this representation system is linked to other more abstract ones. These examples will then serve as a reference for the theoretical framework that follows. We remind the reader that the software described was built for research purposes in the mid 1980s and hence was not refined, especially in terms of interface and screen design, for general use. Nonetheless, more recent teaching experiments involving four sixth-grade classes have shown the software to be quite serviceable in laboratory contexts (Kaput and West, 1993). A software development project underway in the early 1990s (Kaput, Upchurch, and Burke, 1993) embeds this notion of object-based reasoning environments in a fuller set of tools for elementary school mathematics.

A SAMPLE PROPORTIONAL REASONING PROBLEM SITUATION:

Suppose that Noah decides to give 3 umbrellas to each pair of animals coming onto his ark (assuming that each pair will eat one over the next 40 days and 40 nights). If he has 21 umbrellas left, how many more animals should he allow onto the ark?

Using ICE-2 the student first picks two types of icons from a pictorial menu by pointing and clicking, one type to stand for the umbrellas and the other for the animals. Let us choose the triangle to stand for an umbrella and a dog to stand for an animal. We then must input the given quantitative information in the spaces indicated in the screen shown in Figure 8-1.

We will now "grab" the icons from the reservoirs at the bottom of the screen and distribute them into the rectangular array of cells reflecting the intensive quantity 3 umbrellas per 2 animals. We are free to do so any way we choose, but the way we suggest to that minority of students who do not eventually choose it, is to put 3 umbrellas (triangles) and 2 animals (dogs) into each cell until we exhaust 21 umbrellas. We are also free to do this in any order, putting 3 umbrellas and 2 animals into one cell before moving on to the next; or we can distribute 3 umbrellas per cell until we use up 21 umbrellas before going on to accompany each triplet with 2 animals; or we may use a mixed distribution strategy. We can also adjust our distribution by using the "fix" option. Icon-objects moved out of the reservoirs leave gray shadows of their former selves. At each step in the process, the computer reports how many icons of each type have been removed from the reservoirs. In Figure 8-2, we show the results of a successful set of actions, which includes the numerical results displayed in inverse video.

In our teaching experiments, we preceded this proportional reasoning

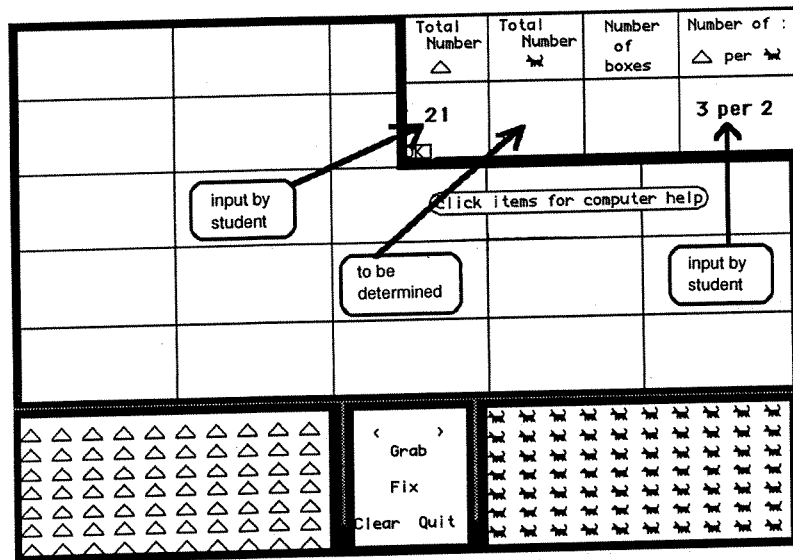


Figure 8-1 A starting-point screen.

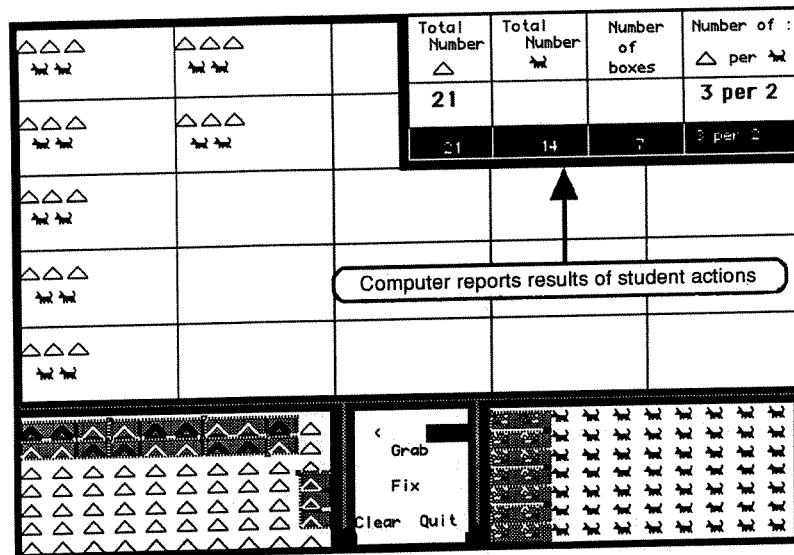


Figure 8-2 Screen illustrating a complete solution.

experience with concrete activities involving simpler multiplication and division with single sets of icons in a very similar environment we call ICE-1. Such problems might involve, for example, determining how many sets of 3 umbrellas could be formed from a set of 21 umbrellas. This can be regarded as a subprocedure of the larger procedure worked out concretely above.

The Boxes Strategy

The majority of sixth grade students with whom we have worked are able to abstract from the above activity a more general two-step strategy for solving such problems based on the cellular distribution of the icons. Indeed, we have seen this strategy appear spontaneously in significant numbers of students who have not been taught it explicitly (Kaput and West, 1993; West et al., 1989), even in considerably younger children. The “boxes strategy” (so named because the students are wont to refer to the rectangular cells as “boxes”) consists of first dividing the given quantity by the number of items per box and then multiplying the resulting number of boxes by the number per box of the second (unknown) quantity. In the above problem this boils down to the following steps:

1. $(21 \text{ umbrellas}) / (3 \text{ umbrellas/box}) = 7 \text{ boxes}$
2. $(7 \text{ boxes}) * (2 \text{ animals/box}) = 14 \text{ animals}$

(Note that the computer reports the number of boxes used as the student progresses through the solution.) As discussed in Kaput and West (1993); West, et al., 1989, and below, this strategy is much more transparent than the standard school-taught strategy for missing value proportion problems, where one sets up and then solves an algebraic equation representing the proportion. That can be learned and applied on a mnemonic basis, and actually functions cognitively as a black box. Virtually no students can explain why they “cross multiply and divide” to solve a proportion equation.

Linking to More Abstract Representations

Actions on sets of objects intended to enact a proportional relationship can be linked to increasingly abstract representational forms: first, a table of data—which records the numerosity of the sets—second, a coordinate graph—which displays the the numerosities and the relationship between them in another way—and third, various forms of algebraic equations to display these in yet another way. Pictured in Figure 8-3 is a screen where all four representations are shown. The picture needs considerable explication, and a student would seldom actually be confronted with all four representations simultaneously. Indeed, the process of building familiarity

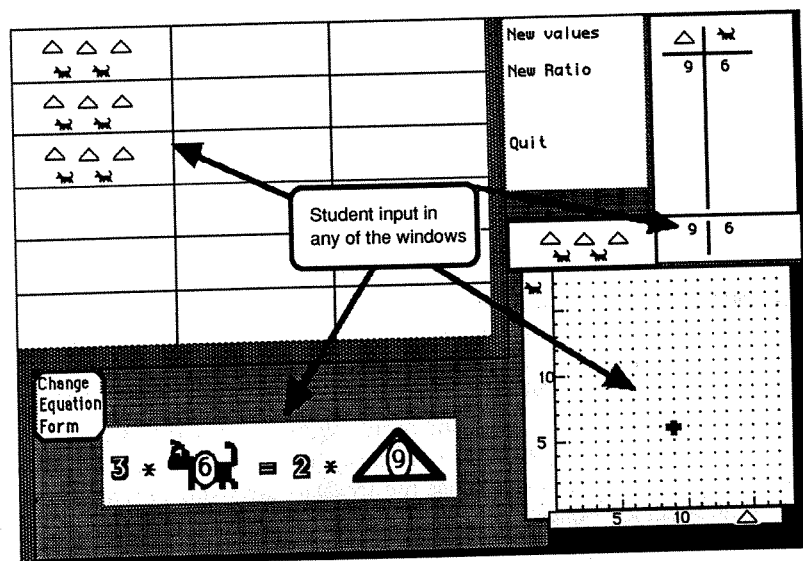


Figure 8-3 Screen illustrating all four representations.

with each of the representations involved requires months, if not years. (Note that by pointing and clicking the mouse button, the student can turn any window on or off as desired—once it has been introduced, of course).

The student is introduced to the new representations one at a time, beginning with the table of data—which is initially used as a way of accumulating an historical record of various iconically computed solutions to problems all sharing the same underlying ratio. Then, with the table familiar as a recipient of previously performed actions on objects, the student can then move to act directly on the table by inputting potential solutions as numbers in the table (actually, in the space below the table since the table itself is reserved for recording accurate solutions). With the actions at least partially abstracted and internalized, inputting numbers in the table amounts to an abbreviation of the previous activity on objects. The student can then view the proposed solution in the icon window if desired.

Similarly, the pattern of introduction is repeated for the coordinate graph, which is introduced as a means for displaying the data accumulated in a table and then can be used as a direct recipient of inputs itself, with the option of viewing the inputs in any of the other representations available. Not shown is an intermediate version of the coordinate graph that ties more closely to the icon representation by stacking sets of icons on the respective axes instead of using numbers.

The algebraic representation has not been fully tested. It is more com-

plex than the others, in part because several equivalent forms of equations can be used, either singly or in pairs. The numbers in ovals are the variables and change in correspondence either with changes made elsewhere or as direct inputs, whereas the other numbers are fixed. For an unbalanced “equation” (corresponding to an asymmetric set of icons, a number pair that does not fit into the table, or a point “off the 2/3 line” in the graph), an appropriately directed inequality sign replaces the equality sign. It is perhaps apparent to the reader, as with many students, that the previously introduced representations act as stabilizing referents for the newer ones.

Two types of actions are supported in the various representations; one type is “natural” to that representation, for example, manipulating objects, typing numbers in the table, clicking on points in the coordinate graph, or typing numbers into the dynamic variables of equations. The other, not shown here, is simple incrementing from a command independent of the representations (we have used simple buttons labeled “MORE” and “FEWER”), where the variables, numbers of umbrellas and numbers of animals, can be incremented separately or simultaneously, with the result viewed in any of the representations as desired.

Theoretical Framework

We now turn to describing a larger view of mathematical representation systems in which the previous work can be interpreted.

A Representational Perspective on Technology and School Mathematics

We find it useful to posit two sources of structure in experience:

1. Mental conceptions, the organizing structures of mind that are used to select, parse and configure the ongoing flow of experience.
2. Materially realizable notation systems (a) that contribute to the structuring of experience by providing means for forming the conceptions, and (b) that support communication of conceptions among individuals.

Notation systems are instantiated within particular media, such as paper-pencil, computer screen, sound waves, and so on. Most school mathematics utilizes historically received, publicly shared notation systems, but notation systems can also consist of personal, idiosyncratic marks.

Mathematical notation systems are usually employed in combination with one another and with natural language. One important way of using notation systems in combination is to use one (A) to represent another (B), as when one uses a coordinate graph to represent a set of ordered pairs. When a notation system is used in this way, we refer to B as a *referent*

of A and the ensemble A, B and the correspondence between them as a *representation system*. When the notation system (A) is used to represent some aspect of a nonmathematical situation (C), for example, the distribution of umbrellas on the deck of an ark, we refer to it as a (mathematical) *model* of that situation. Again, we refer to C as a *referent* of A. In both cases, A has (relative) semantic content provided by the referential relationship. Further explication of these ideas can be found in Kaput (1987, 1989, 1991, 1992).

One reason we pay close attention to the roles of notation systems is the almost infinite representational plasticity of electronic media. We can create notations of almost any variety, in some cases reflecting abstract structures or ephemeral and transient strategies, and in other cases, as with the icon-objects, capturing in the computer medium the very common experiences of manipulating objects. Furthermore, we can link these notations in almost any way we choose creating notational bridges or ramps from the students concrete experience to the ever more abstract objects and relationships of more advanced mathematics.

From our representational perspective, one can characterize three forms of school mathematical activities:

1. Syntactically constrained transformations within a particular notation system (without external referential semantic content).
2. Translations between notation systems, including the coordination of actions across notation systems.
3. Construction and testing of mathematical models.

Currently, the first type of activity strongly dominates at all levels. For example, learning the arithmetic of pure number amounts to learning the syntax of the usual base-ten Arabic numeration system; the bulk of standard secondary school algebra amounts to learning the syntax of transformations of the character strings of algebra. On the other hand, learning the mathematics of quantity amounts to type (3) activity. Quantities, which include numbers and their units, constitute a notation system combining pure numbers and natural language fragments that is an especially fruitful source of models of situations, which act as referents. Furthermore, ICE-2 can be regarded as a concrete representation of the discrete quantity notation system.

Furthermore, unlike syntactically constrained operations on numerals in some algorithm, operations on quantities can be thought of as *semantically* constrained, by students' understanding of their situational referents. In these terms, one can characterize our general approach as building referential meaning for mathematics—either across notation systems as we extend or translate from one to another, or with nonmathematical situations.

There is a fourth and very important kind of long-term learning deserving of mention here. It involves the consolidation or crystallization of relationships and processes into conceptual objects or “cognitive entities”

(Greeno, 1983) that can then be used in relationships or processes at a higher level of organization. Examples of this kind of “vertical” mathematical growth include the act of counting leading to (whole) numbers as objects (Steffe, Cobb, and von Glasersfeld, 1988), taking part-of leading to fraction numbers (Kieren, Nelson, and Smith, 1985), functions as rules for transforming numbers becoming objects that can then be further operated on, for example, added or differentiated (Harel and Kaput, 1991). As we will see, notation systems can facilitate this “entification” or “encapsulation” process.

Interactions Between Conceptions and Notations

An ongoing objective of our research has been to understand how notation systems interact with our mental apparatus to extend mathematical understanding and competence. But, as always, the devil is in the details. To help conquer the devil, we will elaborate the representational perspective outlined before to include the processes of interaction that seem to be at the heart of learning and using notation systems, as depicted in Figure 8-4.

I suggest that we build and elaborate our mental structures in cyclical processes that go in opposite directions. The upward arrow depicts two types of processes: (1) deliberate interpretation (or “reading”), and (2) the more passive, less consciously controlled and less serially organized processes of having mental phenomena *evoked* by the physical symbols. The arrow pointing downward also depicts two types of processes: (1) the act of projecting mental contents onto existing symbols, and (2) the act of producing new ones (“writing”), which includes the physical elaboration

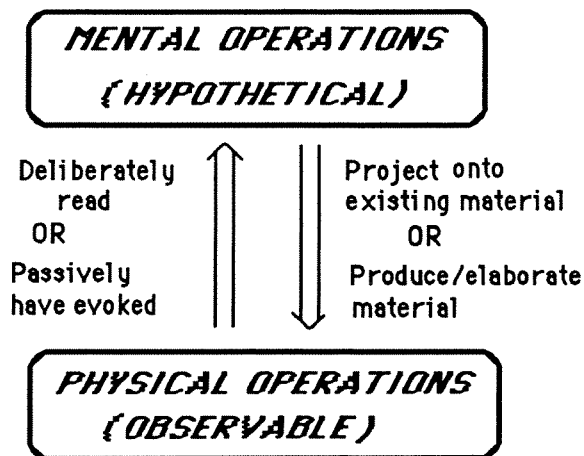


Figure 8-4 Relations between physical and mental structures.

of existing ones. One can also regard the vertical dimension of this diagram as depicting a form of reference, wherein the material notation at the bottom “represents” the conception. This is sometimes referred to, especially in the Piagetian literature, as the signifier-signified relationship (Vergnaud, 1987). It also matches the framework used by Saussure early in this century (Saussure, 1959).

Projections occur in reading and evoking as part of the underlying cyclical processes matching percepts and concepts (as described by Grossberg, 1980). I nonetheless distinguish downward-oriented projections from upward-oriented interpretive acts on the basis of the objective of the process, on where it gets its major impetus. In the downward-oriented case, one has cognitive contents that one seeks to externalize for purposes of communication or testing for viability. Upward-oriented processes are based on an intent to use some existing physical material to assist one’s thinking.

Projections need not amount to writing in the usual sense, but include acting on virtually any kind of physical system or apparatus, where the syntax of the system itself (as well as the medium in which it is instantiated) determines the kind of “writing” that it can accept. In each of the computer-instantiated notation systems described earlier, we see different forms of writing, ranging from standard numerical input to the manipulation of screen objects. We also see here an illustration of a critical difference between a notation system that is computer instantiated versus one that is instantiated in the paper-pencil or physical objects media. By instantiating the syntax of the notation system directly in the computer medium, much more constraint can be built into acts of writing than usually appears in, say, physical manipulatives. A physical objects version of our icon-based notation system would require obedience to a substantially larger set of written rules to accomplish the same level of constraint, to say nothing of the abilities to keep a record of actions taken or link these to another system. (For further discussion of constraints and supports in the computational medium, see Kaput, 1992.)

We now wish to extend this framework to help describe representation acts involving more than one notation system, that is, type (2) and (3) activities. Figure 8-5 helps describe a horizontal dimension of reference—which, as a species of representation, is quite different from the vertical signifier-signified one. Here, I intend to express relationships between notation A and referent B where each (and perhaps even the correspondence) is expressible in material form—but where *the actual referential relationship exists only as mental operations of a person for whom the notations are interpretable*. Such reference exists only by way of composite actions that “pass through” the subjective mental world—as combined acts of interpretation, mental operation, and projection to a physical display—it does not exist apart from the actions of interpretants, although, as the bottom of the diagram suggests, members of a consensual community may

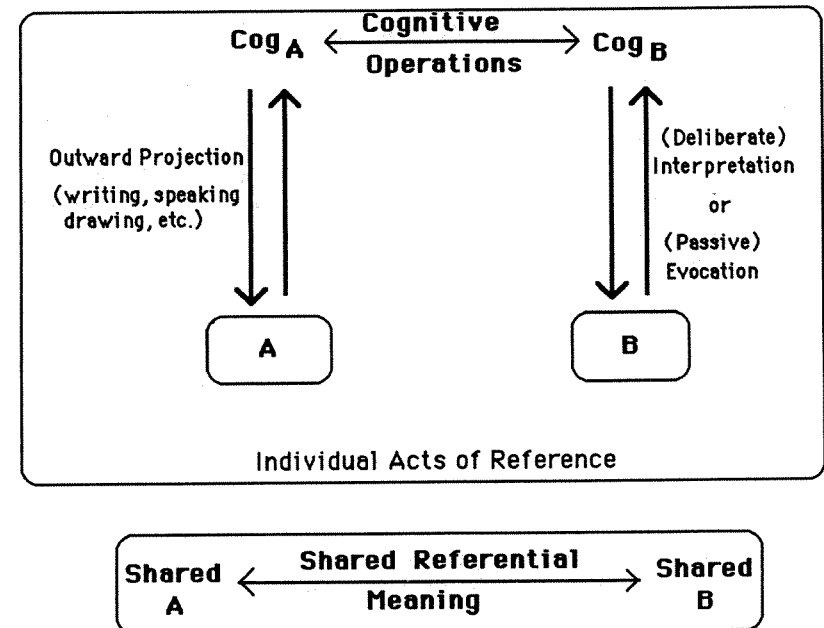


Figure 8-5 A horizontal dimension of reference.

share the referential relationship in the sense of being able to generate the interpretive acts as needed.

The items A and B depict physical observables—for example, an ordered pair in a table A that refers (in the sense just described) to a set of screen objects in a rectangular array of cells B as discussed previously. Of course the reference in this case is based on the numerosity of the two sets of icons involved, so the mental operations in this case include counting the respective sets of objects. Note that *the directionality of the reference in general depends on the cognitive operations involved*, which in turn depend on the context, and hence the direction is not fixed. At one time B may act as a referent for A (e.g., as above, the pair A acting to represent the numerosities of B), and at another the situation may be reversed. The numerals may be given or constructed in advance and one counts out a set of objects to represent the ordered pair of numerals. I suggest that experience with such two-way reference is at the heart of developing competence with multiple representations. I further wish to emphasize that the essence of our work is depicted in the upper levels of these diagrams, where cognitive operations and elaborations, that is, learning and thinking, take place. A fuller discussion of these matters can be found in Kaput, 1991.

conceptualize, respectively, the two (extensive) quantities as cognitive units. The prior knowledge about "pairs" is likely to contribute more to this unit formation than prior knowledge about three umbrellas, which presumably did not exist, and needs to be constructed *de novo*.

Similarly, building the cognitive structure that embodies the "per relation" between these units that constitutes the (intensive) quantity 3 umbrellas per 2 animals is affected by the semantic structure being assembled in Cog(D). Again, prior knowledge of Noah's ark may contribute to the assimilation of the umbrellas into the understanding of the text-based problem statement (the fact that it was raining), and the prior knowledge of the semantic relation between umbrellas and the "organisms" that use them likewise helps form a connection between the two units that yields a higher order cognitive unit made up of the two other cognitive units.

Role of the Features of B—The Rectangular Array of Cells

The "mapping" of the quantitative structure selected and abstracted from Cog(D) to a conceptualization Cog(B) based on the rectangular array of cells B amounts to an integration of cognitive structures. Somehow, the semantic links that help form both the lower order cognitive units, 2 animals and 3 umbrellas, as well as the higher order unit, 3 umbrellas per 2 animals, must be integrated with the structure of the array—assuming that the initial connection between icons and entities in the story has already been chosen (the lowest order entities).

We have seen instances when this integration, or mapping, was subverted by an inappropriate choice of icons, especially in the single icon environment ICE-1 used for posing and solving multiplication and division problems (Kaput, 1988). For example, this occurred most frequently when D involved one entity, say, newspapers, whose semantic relation with the other entity was containment, as with pages. Here in ICE-1, the appropriate choice would be for an icon to stand for a page and the cell to stand for the containing entity, namely, a newspaper. If the student chose an icon, say, a rectangle, to stand for the *containing* entity (newspaper), a breakdown of the solution process followed, and, in some cases, total confusion resulted that was difficult to recover from. This illustrates the strength and importance of the Cog(D) and Cog(B) linkages. We have also seen solution breakdown in ICE-2 when students made capricious choice of the two icons that inadvertently implied a link between the objects being represented that was not part of the given information and that came to influence student actions later in the process.

We have also observed cases of students whose Cog(D) is so poorly structured that they choose an icon to stand for the entity in the problem situation D that appears most prominent to them, for example, picking a person-icon to stand for Noah. In the case at hand, the icon-object relations are more straightforward, although the lack of an umbrella icon forces a choice of a more abstractly related icon. In response to the need for students to be clear about what is standing for what, a new version of

the software (not shown) requires the student to state (in text) what each icon chosen stands for.

Let us now assume icons have been successfully chosen and turn our attention to the specific characteristics of the array and actions on icons within it that seem to be related to the cognitive processes involved in solving the problem. You will notice in Figure 8-1, the small caption "click items for computer help." This appears after the student has entered the given information and can, among other things, help with the icon grabbing and distributing process. In particular, one option assists with the grabbing process by automatically taking either 2 animals or 3 umbrellas when the user chooses one or the other to be moved upward from the reservoirs to the cells. This is a physical instantiation of the first-level cognitive entity formation.

A second and major feature is inherent in the role of the cells as containers for the two sets of icons. The semantic relations between the two types of objects being modeled can vary almost without bound: tight association (4 tires per car), containment (4 candies per bag), arbitrary association (4 ships per 5 antelopes), price (4 muffins per 3 dollars), and so on. The perceptual feature of the array that corresponds to any such relation between the two types of objects is mutual containment and contiguity within the same cell. We have not implemented, but could easily have, a feature that enhances the grabbing action so that a single "grabbing" action would result in isolating and joining both types of objects simultaneously, for example, 3 umbrellas and 2 animals would be moved to a cell simultaneously. This would supplant the separate grabbing actions that a student must now perform. With Piaget, we have assumed that "cognitive entification" occurs as the result of reflection of internalized actions to a new organizational level ("reflective abstraction"). Hence we require the student to perform actions, in this case grouping actions, to initiate the needed entifications.

Alternative organizations of intensive quantities are sometimes appropriate, for example, when one needs to keep the aggregate sets of objects separate, rather than mixing them homogeneously. For example, one can imagine that a store sells 3 medium-size sweatshirts for every 2 small-size ones. In buying an inventory of sweatshirts, one would order sets of each where the aggregate subtotals would maintain the 3:2 ratio, but with an assumption that the groups would remain separate. There is yet another organization of the quantities and the objects associated with an intensive quantity such as 3 umbrellas per 2 animals, a chunking of the whole set of 21 umbrellas with the whole set of 14 animals. This organization is only weakly reflected within the array of ICE-2 because the sets of icons remain in 3:2 subsets inside separate cells. A stronger feature would allow the objects to migrate together into subsets of 21 and 14 elements inside a larger containing set. However, this level of organization is necessarily embodied in the numerical representation of the total number of umbrellas and animals that have been moved up into the array. By counting the

respective elements in the separate cells, one is treating them as subsets of a larger set, and the resulting numerosity of the respective two subsets likewise addresses this new level of organization. Thus this level of organization appears in the the window in inverse video—see Figure 8-2.

An interesting trade-off exists between the cellular organization and the larger subset organization, whereby the former emphasizes a homogeneity of the distribution of umbrellas and animals and the latter emphasizes their respective collectivities. There is an underlying homogeneity in the situation being modeled in the example at hand—Noah gives 3 umbrellas to each pair of animals appearing at dockside. Furthermore, the incrementing/distributing action in the situation parallels that of the ICE-2 model. In other situations, the underlying actions might not be as parallel and the homogeneity might be more problematic, as when one is modeling a neighborhood where, overall, there are 3 cars for every 2 houses. Here, homogeneity probably does not hold, yet it is customary to use an intensive quantity in this way to describe the “density” of cars in the neighborhood. We have developed another environment that supports a sampling activity (see Kaput and Pattison-Gordon, 1987) which addresses the issue of homogeneity even more directly (see also Schwartz, 1988, who discusses local vs. global application of intensive quantities). In yet another situation, the two types of objects might be “clumped together” separately—for example, when one is balancing sets of A-objects with sets of B-objects on a scale, where 3 A-objects weigh the same as 2 B-objects. Here, the model closest to the situation would involve maintaining the two types of objects as separate sets.

While the underlying mathematical model for these different types of situations may be the same, the appropriate intermediate concrete models might vary. The representational plasticity of the computer medium allows us to create different types of intermediate modeling environments with different types of features that require less initial abstraction from the situation being modeled, such as ICE-2 and its variants discussed previously. However, the number of these is not only limited, it should be quite small to avoid a Tower of Babel effect. Nonetheless, we have concluded that *no single system can capture all the essential aspects of such a complex idea-web as intensive quantity. And in each case there should be some link to the more abstract mathematical models traditionally used*, for example, coordinate graphs, tables, and equations, as in Figure 8-3.

Other Aspects of the Idea-Web of Intensive Quantity

Related to the matter of homogeneity of the objects or material that is to be modeled using intensive quantity is the role of the intensive quantity as a descriptor of an *intensive attribute* of the situation being modeled. Assuming homogeneity, a description of, say, the price per muffin in a bakery, tells nothing of the number of muffins sold or the total amount of money they cost their buyers. The price applies to all sets of muffins as well as it

does to any one of them. Depending on the situation one is attempting to model, especially if some kind of homogeneity is assumed, this role of intensive quantities as global descriptors of intrinsic properties of situations may or may not be important. For example, if Noah were attempting to match many different-sized sets of animals with umbrellas, then his attention would be drawn to the fact that the 3 umbrellas per 2 animals is a general description that applies to all sets, and not merely to a particular one—he would be likely to think of the intensive quantity as a general rate rather than a particular ratio (Kaput and West, 1993; Thompson, 1992, 1993). On the other hand, when focusing on a situation involving only one value of the variables constituting the numbers of umbrellas and animals, this aspect of intensive quantity plays only a small role.

Before closing this discussion of relations between situations and potential models, we wish to emphasize the value of certain representations of intensive quantities in other reasoning processes not yet mentioned. One important example of such a process involves comparing magnitudes of intensive quantities with the same dimension.

Among the notations that we have discussed to date, the representation of an intensive quantity as the slope of a line in a coordinate graph offers the easiest comparison of intensive quantities, because a given intensive quantity, as the slope of a line consisting of an unbounded set of ordered pairs, is visually represented by a single conspicuous thing, the “tilt” of the line. Any other intensive quantity (with the same dimension, of course) can be compared with it by comparing the associated “tilt,” because this tilt is a linearly ordered attribute of lines through their common point, the origin.

Consider the following two problems:

1. Which neighborhood has more cars per house, one where there are 3 cars for every 2 houses or one where there are 5 cars for every 3 houses?
2. Which muffins are more expensive, those that cost 2 dollars for 3 muffins or those that cost 3 dollars for 5 muffins?

Let us now compare the two problems regarding the structure of the prior knowledge involved in their solution (especially in building a Cog (D) as in the Noah’s Ark problem—see Figure 8-6) and the structure of the representation B used in the reasoning process to solve them. In the muffin price comparison, we apply significant prior knowledge at the level of the initial scheme. In particular, most people already have available an intensive quantity to apply in such a situation, called “price,” although it is likely to be in the unit-price form of dollars per item rather than in number of items per number of dollars. Depending on the solver’s flexibility in transforming this prior knowledge into the form needed, this prior knowledge may help or hinder the mapping from Cog(D) to B. The mapping boils down to associating price with a line.

Just as important as the existence of this prior knowledge is the existence of *price-comparison* knowledge, whereby one interprets “more expensive.” In other words, there exists a linear comparison scheme for price as well, which maps onto the tilt-slope comparison in B. Put differently, it helps interpret the differences in observed slope.

All this is meant to contrast with the cars-houses problem, where the prior knowledge is not likely to possess such structure. Number of cars per number of houses is not a quantity that we often deal with, and unlike “price,” we do not have a ready-made word for it. And even less likely is knowledge about comparisons of such quantities, referred to as neighborhood “car density” (using an analogy with a very important type of application of intensive quantities in describing physical densities of various types).

Thus, we see important yet subtle differences in the existence of prior knowledge and its relation to a notation system.

Some Reflections on Computers and Notational Ramps and Bridges

Bridging Notations and Cognition

The Sapir-Whorf hypotheses regarding the influence of language on thought (Whorf, 1956), while contestable and difficult to operationalize in the context of natural language and everyday cognition, appear without question to have merit in the context of the special notations of mathematics. Different notation systems support dramatically different forms of reasoning, although the differences are strongly influenced by interactions between the knowledge structures associated with the notations and the prior knowledge brought to the reasoning. Given the immense complexity of human knowledge structures, such interactions are not easily accounted for. Nonetheless, we can be relatively explicit about the operant features and the knowledge associated with particular notations, both traditional and novel. Figure 8-7 is intended to help summarize our position.

We have been especially interested in using computers to support two forms of referential meaning-making, one involving reference relations between mathematical systems, for example, tables and graphs, and the other between mathematical and nonmathematical ones. The former are less problematic, and instantiating these in computer systems is a well-established use of modern computational power, for example, connecting equations and graphs. This appears in the lower left side of Figure 8-7. Nonetheless, they offer considerable promise by off-loading low level translation activity to the computer so that the student can focus on higher level relationships and action-consequences across representations. Another important contribution they offer is freeing the translation process from

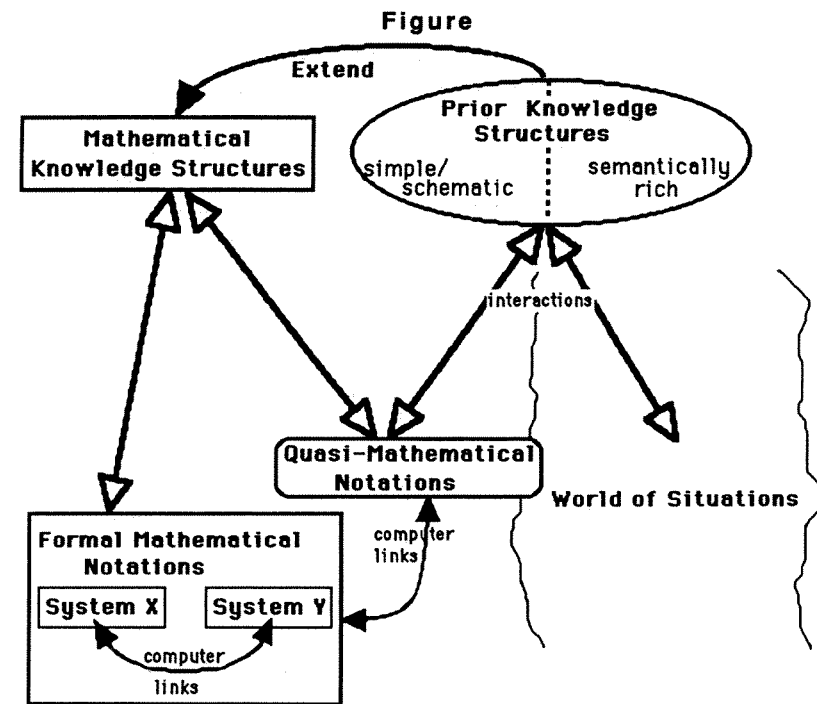


Figure 8-7 Overall framework.

serial, real-time constraints to virtually time-independent translation, so that results of actions across representations can be examined at will, in ways that optimize the development of understanding.

The other source of mathematical meanings is the rich reservoir of “natural” knowledge built from everyday experience, depicted in the upper right side of Figure 8-7. As indicated, we separate this knowledge into two general types, one being the rich, multilayered semantic knowledge of situations, as the knowledge of Noah’s ark, for example. The other type is the somewhat simpler schematic knowledge by which the richer, more complex knowledge is often interpreted. An example includes our knowledge of discrete objects, their counting, grouping, and manipulating. We assume that the interpretation of complex, ambiguous situations proceeds through the use of this kind of simpler knowledge.

Our design of object-based reasoning environments such as ICE-1 and ICE-2 is intended to assist this interpretive process by reflecting this simpler knowledge in intermediate, quasi-mathematical notations, analogous to traditional manipulatives. These notations can then help bridge between the highly economical and efficient formal mathematical notation

systems and the situations of everyday experience that the formal systems are intended to model. As our previous section attempted to illustrate, the interactions between the two types of knowledge (shown in Figure 8-7) are at the heart of the bridging process.

Using physical manipulatives to help teach mathematics has long been accepted as a good idea (Lesh, Post, and Behr, 1987; Post, 1980), especially as a means for embodying the structural features of a mathematical idea or procedure, that is, representing it in a quasi-mathematical notation. But manipulatives have never been widely accepted in practice for three reasons: (1) classroom management difficulties, (2) cognitive management difficulties (students exhaust all their cognitive resources performing actions in one or the other notation systems and have none remaining to track active relations between them, especially at the level of actions rather than merely objects), and (3) the curricular press to computational skill. While the third reason is only indirectly affected by the availability of computers, *cybernetic* manipulatives, used as bridging notations in the computer medium, offer a potential solution to the first two difficulties, especially by assisting the translation process. The reader will notice that manipulatives, such as Dienes Multibase Blocks, either wooden or cybernetic as in Thompson's *Blocks Microworld* (Thompson, 1992) involve only half the scheme outlined above—they do not involve the prior knowledge of semantically rich situations because their focus is on the structure of the usual number notation system rather than on modeling semantically rich situations. Incidentally, the results reported by Thompson (1992) in a careful comparison of traditional versus cybernetic multibase blocks showed significantly superior learning in the computer context except for the weakest students.

In Kaput (in press a) we offer yet another set of reasons for utilizing cybernetic manipulatives. When one acts on physical materials, each action overwrites the previous state, which can seldom be recovered, at least by the student. In a computer environment we can record significant prior states and student actions, and then represent the sequence of states that result—as a “filmstrip” of miniature screens, for example. Such a sequence might be replayed, offered in advance by a teacher to be “debugged,” or otherwise changed. A student would be able to reopen any strip at any frame and resume the action, perhaps to repair or to alter a strategy to yield a branch off the original linear strip. Students could also compare one strip against another, in effect, comparing strategies as visible objects. In this way students' attention and action may be focused at a higher level. In (Kaput, in press a) we also discuss the option of recording the actions as a specialized script. In a sense, the filmstrips and scripts constitute complementary representations of a student's work—filmstrips describe the states while scripts describe the state-transitions. If specific scripts, based on particular actions on particular quantities, could have their particular values replaced by variables, then they also act as algorithms. A whole new option space opens up for building and testing algorithms by generalizing

specific cases. Clearly, the world of cybernetic manipulatives merits serious exploration.

We Need a Theory to Inform Deliberate Construction of Notations by Designers and Students

Our design decisions for bridging notation systems have been mainly of the seat-of-the-pants type, driven by informal pedagogical knowledge. The new design space in mathematics as well as other domains is only newly opened, let alone explored (Norman, 1986). Its fruitful exploration will require better understanding of the structure of students' relevant prior knowledge and how to structure bridging notations so as to optimize the power of that prior knowledge without trapping students in the limitations associated with the very specificity from which it draws its strength. How closely should the notations match the semantic richness of situational knowledge?

The evolution of traditional notations, especially in mathematics (Cajori, 1929a,b), has yielded design decisions biased strongly on the side of generality and syntactical economy—from which they draw their enormous strength. That evolution, of course, took place within the constraints of static media and without any need to ensure learnability by the majority of the population—the notations were created by and for an intellectual elite (Kaput, in press b). *Both factors no longer obtain.* Now we have dynamic media of essentially unlimited plasticity, and we must make important ideas learnable by the large majority of the population. The key challenge seems to be to strike the right balance between abstractness and generality on one hand and concreteness and specificity on the other. Such balance is likely to vary from topic to topic and according to the needs of different learners.

Another set of questions even farther from being answered involves how to enable *students* to create and link their own notations. Building symbols is a major part of what it means to be human, both as a cultural phenomenon and as a matter of individual growth and cognitive function. On purely academic grounds, this skill in exploiting the representational plasticity of new media is of increasing importance and is likely to surpass skill in computation in importance, if it has not already done so. On a cultural level as designers, we are in a position somewhat analogous to the bioengineers relative to evolutionary change. We no longer need to wait for the Blind Watchmaker to fashion our notations. We can be more deliberate, more systematic, but we need more prior knowledge—of how notations function.

Note

The author wishes to thank Judah Schwartz for early and continuing support for and guidance in this work. The author also wishes to thank Clifton Luke, Laurie

Patterson-Gordon, Joel Poholsky, and especially Mary Maxwell West, for their important contributions to the work reported here—it would not have happened without their help. Some of the research reported on has been funded by NSF Grant MDR 885617.

References

- Cajori, F. 1929a. *A history of mathematical notations*. Vol. 1. Notations in elementary mathematics. La Salle, IL: The Open Court Publishing Co.
- . 1929b. *A history of mathematical notations*. Vol. 2. Notations in elementary mathematics. La Salle, IL: The Open Court Publishing Co.
- Greeno, J. 1983. Conceptual entities. In *Mental models*, edited by A. Stevens and D. Gentner. Hillsdale, NJ: Erlbaum.
- Grossberg, S. 1980. How does a brain build a cognitive code? *Psychological Review* 87: 1–51.
- Harel, G., and J. Kaput. 1991. The influence of notation in the development of mathematical ideas. In *Advanced mathematical thinking*, edited by D. Tall and E. Dubinsky. Amsterdam and Boston: Reidel.
- Kaput, J. 1985. *Multiplicative word problems and intensive quantities: An integrated software response*. Technical Report 85–19. Cambridge, MA: Harvard Graduate School of Education, Educational Technology Center.
- . 1987. Toward a theory of symbol use in mathematics. In *Problems of representation in mathematics learning and problem solving*, edited by C. Janvier. Hillsdale, NJ: Erlbaum.
- . 1988. Supporting concrete visual thinking in multiplicative reasoning. In *Focus on learning problems in mathematics*, edited by T. Eisenberg and T. Dreyfus. Special issue on Visualization and Mathematics Education, 35–48.
- . 1989. Linking representations in the symbol system of algebra. In *A research agenda for the teaching and learning of algebra*, edited by C. Kieran and S. Wagner. Reston, VA: National Council of Teachers of Mathematics; Hillsdale, NJ: Erlbaum.
- . 1991. Notations and representations as mediators of constructive processes. In *Constructivism and mathematics education*, edited by E. von Glasersfeld, pp. 53–74. Dordrecht, Netherlands: Kluwer Academic Publishers.
- . 1992. Technology and mathematics education. In *Handbook on research in mathematics teaching and learning*, edited by D. Grouws, pp. 515–56. New York: Macmillan.
- . 1994. The representational roles of technology in connecting mathematics with authentic experience. In *Mathematics as a didactic discipline: The state of the art*, edited by R. Bieler, R. W. Scholz, R. Strasser, and B. Winkelmann. Dordrecht, Netherlands: Kluwer Academic Publishers.
- . In press a. Overcoming physicality and the eternal present: Cybernetic manipulatives. In *Visualization and technology in mathematics education*, edited by R. Sutherland and J. Mason. New York: Springer-Verlag.
- . In press b. Democratizing access to calculus: New routes using old roots. In *Mathematical thinking and problem solving*, edited by A. Schoenfeld. Hillsdale, NJ: Erlbaum.

- Kaput, J., and L. Pattison-Gordon. 1987. *A concrete-to abstract software ramp: Environments for learning multiplication, division, and intensive quantity*. Technical Report 87-8. Cambridge, MA: Harvard Graduate School of Education, Educational Technology Center.
- Kaput, J., and M. West. 1993. Assessing proportion problem complexity. In *The development of multiplicative reasoning in the learning of mathematics*, edited by G. Harel and J. Confrey. Research in Mathematics Education Series. Albany: State University of New York Press.
- Kaput, J., R. Upchurch, and M. Burke. 1993. Integrated tools for elementary mathematics. [Software]. No. Dartmouth: University of Massachusetts at Dartmouth, Department of Mathematics.
- Kaput, J., M. M. Katz, C. Luke, and L. Pattison-Gordon. 1988. Concrete representations for ratio reasoning. In *Proceedings of the 10th Annual Meeting of the PME-NA*, edited by M. Behr, C. Lacampagne, and M. M. Wheeler. DeKalb, Illinois.
- Kieren, T., D. Nelson, and G. Smith. 1985. Graphical algorithms in partitioning tasks. *Journal of Mathematical Behavior* 4: 25–36.
- Lesh, R., T. Post, and M. Behr. 1987. Dienes revisited: Multiple embodiments in computer environments. In *Developments in school mathematics around the world*, edited by I. Wirzup and R. Streit, pp. 647–80. Reston, VA: National Council of Teachers of Mathematics.
- Miera, L. 1991. Explorations of mathematical sense-making: An activity-oriented view of children's use and design of material displays. Ph.D. diss., University of California, Berkeley.
- Norman, D. A. 1986. In Norman, D. & Draper S. (Eds.) *User-centered design*. Hillsdale, NJ: Erlbaum.
- Saussure, F. de. 1959. *Course in general linguistics*. Glasgow: Fontana. Original edition in French, 1916.
- Schwartz, J. 1988. Referent preserving and referent transforming quantities. In *Number Concepts and Operations in the Middle School*, edited by J. Hiebert and M. Behr. Reston, VA: National Council of Teachers of Mathematics; Hillsdale, NJ: Erlbaum.
- Steffe, L.P., P. Cobb, and E. von Glasersfeld. 1988. *Construction of arithmetical meanings and strategies*. New York: Springer-Verlag.
- Thompson, P. 1992. Notations, conventions and constraints: Contributions to effective uses of concrete materials in elementary mathematics. *Journal for Research in Mathematics Education* 23 (2): 123–47.
- . 1993. The development of the concept of speed and its relationship to concepts of rate. In *The development of multiplicative reasoning in the learning of mathematics*, edited by G. Harel and J. Confrey. Research in Mathematics Education Series. Albany: State University of New York Press.
- Vergnaud, G. 1983. Multiplicative structures. In *Acquisition of mathematics concepts and processes*, edited by R. Lesh and M. Landau. New York: Academic Press.
- . 1987. Conclusion. In *Problems of representation in mathematics learning and problem solving*, edited by C. Janvier. Hillsdale, NJ: Erlbaum.
- . 1988. Multiplicative structures. In *Number concepts and operations in the middle grades*, edited by J. Hiebert and M. Behr. Reston, VA: National Council of Teachers of Mathematics.
- West, M. M., C. Luke, J. Poholsky, L. Pattison-Gordon, S. Turner, and J. Kaput.

1989. Additive strategies in proportional reasoning: Results of a teaching experiment using concrete representations. In *Proceedings of the 11th Annual Meeting of the PME-NA*, edited by G. Goldin, C. Maher, and T. Purdy. Rutgers, NJ.

Whorf, B. 1956. *Language, thought, and reality*, edited by J. B. Carroll. Cambridge, MA: MIT Press.