Chapter 19

THE STRAIGHT AND NARROW: LINEAR DEPENDENCE IN THE SCIENCES

Moreover the simplest derived geometrical concepts, to which here belong especially the line and the plane, correspond to those which suggest themselves most naturally from the logical standpoint.

(Hermann Weyl)

A BASIC QUESTION in almost any science is this: how does one thing depend on another? In physics, for example, we might ask how an object’s position depends on time, or how a current depends on voltage. In chemistry, we might ask how the rate of a reaction depends on temperature, or how the reactivity of a metal with an acid depends on the pH of the acid. And in biology, we might ask how the metabolism of an animal depends on the amount of some hormone in its blood, or how the growth of a plant depends on the amount of rainfall it receives. These few examples could be multiplied almost without end. Sometimes, how one thing depends on another is quite complicated, but in this chapter we are interested in a very simple kind of dependence. Mathematically, we say that one variable depends on another variable, or that one variable is a function of another variable. A mathematical function is just a way of specifying the dependence. Again, the functional dependence may be simple or it may be complicated, and in this chapter we’ll look at the simplest case possible: the linear function, otherwise known as a straight line.

§1. Basic Ideas

To understand what linear dependence means, let’s start with an example that is familiar to most people, namely, working for an hourly wage. If you are working at a job that pays $12.00 per hour, then you’ll make $12.00 working for one hour, $24.00 working for two hours, $36.00 working for three hours, and so on. We can summarize all of the particular cases in a single statement simply by saying that the amount of money you make is equal to the number of hours you work multiplied by $12.00 per hour. This can be written in the form of an equation,
The amount of money you make is said to be directly proportional to the amount of time you work. “Directly proportional” simply means that one quantity is equal to another quantity times some constant number (called the proportionality constant). In our example, the proportionality constant is $12.00 per hour. Of course, the proportionality constant might be different; for example, you might get a raise to $14.00 per hour. The amount of money you make is still linearly dependent on the number of hours you work, but you make more money now (after your raise) for the same amount of time worked. The proportionality constant, in this case, is a measure of how rapidly you accumulate money. We now have two different ways to say the same thing. A variable that depends linearly on another variable is, by definition, directly proportional to that variable; these two relationships are identical. What meaning does the word “linear” have in this context, and why do we use it as a synonym for direct proportionality? The word “linear” comes from the word “line.” To see why that’s appropriate, look at the graph in Figure 27. The amount of money you make is plotted on the vertical axis versus the number of hours you work plotted on the horizontal axis. As you see in Figure 27, the graph of a direct proportionality relationship is in fact a straight line. So, we call the relationship linear.

The solid line in Figure 27 is a graph of the money you make before your raise, while the dotted line is after your raise. They are both straight lines, but the line for $14.00 per hour has a steeper angle with the horizontal. The angle that a straight line makes with the horizontal axis always depends on the proportionality constant of the graphed relationship. The larger rate of pay having the steeper angle is no accident. For this reason, the proportionality constant is called the slope of the graph (e.g., the slope of the steeper line is $14.00 per hour).

All of these ideas are quite general. Instead of the money you make, think of some arbitrary quantity $y$; instead of the hours you work, think of another arbitrary quantity $x$; and instead of your wages, think of any arbitrary constant $m$. Instead of the previous equation, we now have

$$y = mx.$$ 

This may look more abstract, but $y = mx$ really isn’t any more complicated than our simple example. The advantage of this more abstract version is that these symbols can now stand for anything we want. Let’s illustrate the point with one more simple real-life example of a linear relationship. If you are traveling on the highway in a car that gets a gas mileage of 42 miles per gallon, the distance you travel is directly proportional to the
amount of gasoline you use. In this example, \( y \) is equal to the distance you go, \( x \) is equal to the number of gallons of gasoline used, and \( m = 42 \) miles per gallon.

We can add a constant to our linear equation and still have a straight line with the same slope. Instead of just the money you make, for example, you might be more interested in your total savings. In that case, you add the (linearly increasing) money you are earning onto the amount of money you had to start with. Graphically, the effect of adding a constant is to shift the entire straight line vertically upward or downward on the graph. Our modified linear equation might look like

\[
y = mx + b,
\]

where \( b \) stands for the value of the constant we’re adding. This constant is sometimes called the \( y \)-intercept of the graph (because \( y=b \) when \( x=0 \), which is the \( y \)-axis).
Regardless of the particular value of $m$, our main interest may simply be that $y$ is in fact directly proportional to $x$. In other words, $y$ varies linearly with $x$, and this linear dependence is often the crucial information concerning some scientific phenomenon; the details of the proportionality constant may be less important to us. Scientists emphasize the importance of the functional dependence itself by using the symbol “$\propto$” which means “is proportional to.” So, instead of using an equation, we can simply write

$$y \propto x$$

to express the fundamental idea we’re interested in. To actually find numbers for $y$ that correspond to numbers for $x$, however, we obviously need the proportionality constant.

§2. EXAMPLES OF LINEAR VARIATION IN THE SCIENCES

Our primary motivation for discussing linear dependence is its usefulness and widespread application in the sciences. The property that makes linearity so useful is that it’s the simplest functional dependence that two variables can have. (Even if there is no dependence, meaning one variable remains constant while the other changes, this too is a straight line, with a slope of zero.) A process or phenomenon that is governed by a linear relationship is easy to analyze and to understand. Fortunately, many such linear relationships are found in nature.

Constant Velocity

A simple example is motion with a constant velocity. Constant velocity implies a steady speed and direction, moving the same distance in each equal time interval. (If you move 25 meters during each second, for example, you have a constant velocity of 25 m/s; this is about 56 miles/hour.) The distance you travel is directly proportional to the time you’ve been traveling. The constant of proportionality in this case is the velocity. So,

$$\text{distance} = (25 \text{ m/s})(\text{time})$$

or, more generally,

$$x = vt,$$
where $x$ stands for the distance, $t$ stands for the time, and $v$ stands for the velocity. Graphs of distance versus time are given in Figure 28 for velocities of 25 m/s, 10 m/s, and 35 m/s, in order to illustrate once again the relationship between the appearance of the line and the numerical value of the slope (i.e., velocity).

In both our first example (involving money) and our last example (involving distance), the relevant quantity changes linearly with time. Put differently, both cases are concerned with a constant rate of change for some quantity. A constant rate of change is a fairly common application of the idea of linear variation in the sciences, and also in more general real-life situations. Another example might be a steady rainfall, in which the water level in a rain gauge increases linearly with time. But not all linear variations have to do with time rates of change, as our next examples show.
Density

Consider the relationship between mass, volume, and density. By definition, the density of a substance is its mass per unit volume. In other words, the density of an object is the mass of this object divided by its volume. This relationship can be written as

\[ \rho = \frac{m}{V}, \]

where \( \rho \) is the density, \( m \) is the mass, and \( V \) is the volume. (\( \rho \) is the Greek letter rho.) Multiplying both sides of the equation by \( V \), we have the equivalent form

\[ m = \rho V. \]

For any particular substance (characterized by some density), the mass varies linearly with the volume. In a graph of mass versus volume, the slope of the straight line would be equal to the density of the substance. We see that the mass is directly proportional to the volume of an object, while the density is a property of the substance that the object is made of (and is unaffected by the size or shape). Of course, for a set of objects having the same volume, but made of different materials, the mass of each object is directly proportional to its density. Incidentally, it’s interesting to note that, based on the material discussed in chapter 16, the mass does not vary linearly with the characteristic size or with the surface area of an object.

Ideal Gas Law

Let’s now look at the ideal gas law from chemistry. As an equation, the ideal gas law can be written as

\[ PV = nRT, \]

where \( P \) is the pressure a gas exerts, \( V \) is the volume of the container the gas is in, \( n \) represents the amount of gas, \( T \) is the temperature of the gas, and \( R \) is a constant (known as the universal gas constant). Suppose we keep the volume of the container and the amount of gas fixed. The pressure of the gas is then directly proportional to its temperature. \( P \) varies linearly with \( T \); the proportionality constant in this case is \( nR/V \). A variety of linear dependences are implied by the ideal gas law. If we keep the pressure constant, for example, the volume varies linearly with the temperature for a fixed amount of gas. If both the pressure and the tempera-
ture are held constant, then the volume is directly proportional to the amount of gas \((\text{slope} = \frac{RT}{P})\). A \(V \propto n\) dependence is, of course, quite sensible if you think about it. But all of these simple linear relationships are predicated on the rest of the variables in the equation being held constant. If more than two quantities can vary at once, then we lose the simple linearity.

**Hooke’s Law**

Another example, from physics, is known as Hooke’s law. Hooke’s law is usually introduced in connection with the force that a spring exerts when you stretch it (or compress it). The law states that the force exerted on an object is directly proportional to the distance through which the object is displaced. For a stretched spring, this means that the spring pulls back with a force proportional to the distance through which it is stretched. If you pull twice as far, the spring pulls back twice as hard, and so on. As an equation, Hooke’s law is written as

\[ F = -kx, \]

where \(F\) is the force, \(x\) is the distance stretched (or compressed), and \(k\) is the proportionality constant. The minus sign is there because the spring pulls in the direction opposite to the direction of the displacement (a force acting like this is called a restoring force). What meaning does the proportionality constant \(k\) have in this case? If \(k\) is large, then a small displacement results in a large force (and the opposite is also true). Some thought then reveals that the proportionality constant \(k\) in Hooke’s law is a measure of how stiff the spring is. One reason why Hooke’s law forces are so interesting is that many different physical systems are governed by forces that (at least approximately) have this form. A diatomic molecule, a guitar string, an atom in a solid, a pendulum, and a floating object that bobs up-and-down, are all examples of systems having restoring forces linearly proportional to displacements from equilibrium (equilibrium is defined as the position where the force is zero). Another reason Hooke’s law forces are interesting is that such forces, being linear, are simple. Because the forces are so simple, we’re able to analyze their effects and predict the motions they cause (these motions turn out to be periodic oscillations).

**Other Examples**

Next, let’s look at an example from biology. The amount of oxygen consumed by an organism is directly proportional to the amount of energy it uses in metabolic activities. As a particular example of this, the stomach
uses energy to secrete digestive acids. A graph of the oxygen uptake by the stomach versus the rate of stomach acid secretion is a straight line. Approximate linearity (see §3) is often useful in biology. The flow rate of blood through the circulatory system, for example, is approximately proportional to the pressure drop in the system (see chapter 6).

Finally, we'll consider one more example from chemistry. The boiling point and freezing point of a liquid changes if something is dissolved in the liquid. The boiling point gets higher and the freezing point gets lower. This lowering of the freezing point is familiar to everyone who has used salt to melt ice from sidewalks and roadways in the winter. The amount by which the temperature (of the freezing or boiling point) changes is directly proportional to the concentration of the solution. The constant of proportionality, in this case, depends on the identity of the components making up the solution.

§3. Approximate Linearity

As we see in these examples, many phenomena in the sciences are linear, which is one reason why linearity is important. Another reason is that certain phenomena, which are not really linear, can still be considered approximately linear (at least for some range of the variables). In other words, they are almost linear. (Approximate models are discussed more thoroughly in chapter 6.) In such cases, we can exploit the inherent simplicity of linear dependence in our analysis. As a simple example of approximate linearity, reconsider the gas mileage of your car. We said in section 1 that the distance you drive is linearly proportional to the amount of gasoline you use. But this isn’t quite right. Some of your journey might be uphill, where your gas mileage is lower; highway driving gives you better mileage than city driving; and so on. The “constant” of proportionality actually varies a bit as driving conditions change. The relationship is only approximately linear. We’ve already seen a similar example from science, namely the relationship between blood flow rate and pressure drop in the circulatory system. Once again, the proportionality constant (resistance to blood flow) varies somewhat with the pressure drop, instead of being a genuine constant.

An interesting example of approximate linearity is shown in Figure 29, where atomic weights of elements are graphed versus their atomic numbers. The atomic number of an element is the number of protons the element has. The atomic weight depends on the number of neutrons and protons together (they have similar masses). The number of protons and the number of neutrons are roughly equal in each element, but not exactly equal. For this reason, the atomic weight is roughly proportional to the
Figure 29. Graph of the relationship between atomic weight and atomic number in the elements, showing an example of approximate linearity. Along with the small amount of scatter, notice the upward curve at high atomic number (seen more easily with a straightedge or by looking at the plot along a glancing angle).

atomic number (with \( m \) approximately 2), but not exactly proportional. This relationship had interesting historical consequences. When the periodic table was first worked out (see chapter 2), the periodicity was in the atomic weights (which were known experimentally) rather than in the atomic numbers (which had not yet been invented as a concept; protons, neutrons, and nuclei would not be discovered for many years).

Our last example, from biomedical work, has important public policy ramifications. When an organism (e.g., a human) is exposed to some chemical (such as a medicine or a toxin), and the chemical has an effect, this effect is called a response to the chemical. The amount of the chemical to which the organism is exposed is called the dose. We can certainly expect some relationship between the dose and the response. If the response is quantified in some way, we can make a graph of the response versus the dose, sometimes called a dose-response curve. It’s not uncommon for the dose-response curve to be approximately linear, espe-
Figure 30. Graph of a clearly nonlinear relationship (a sine curve), along with a straight line that is an excellent approximation for part of the curve.

Especially if the dose is not too high or too low. Questions about the dose range over which we can correctly assume the linear response approximation are important. For example, the toxicity of chemicals and radiation studied at relatively high doses is assumed to be linear down to low doses, where empirical studies are difficult. If the linear approximation breaks down at low doses, the danger of exposure might be underestimated or overestimated.

Many mathematical relationships that are not linear can also be considered approximately linear over some range. In Figure 30, we see a graph of a relationship that is decidedly nonlinear (it’s actually a trigonometric relationship known as a sine curve). On the same graph, we plot a straight line. The straight line and the sine curve are virtually identical up to about 20 degrees. Most curves are nearly linear over some range of variables that is small enough, giving us an insight into why approximate linearity is common in nature. The opposite is also true: Most linear relationships break down if the range of variables is too extreme. The ideal gas law, for example, breaks down when the pressure is very high or the temperature
is very low (see chapter 6; the density becomes high in both of these cases, and it’s not surprising that the ideal gas law breaks down with the gas on the verge of becoming a liquid). Hooke’s law also breaks down if the displacements become too large (which again is not too surprising, since you know that you can’t stretch a spring indefinitely). These breakdowns do not detract from the usefulness or wide applicability of linear relationships. Any relationship in science is only valid within some proper domain of applicability.

Linear relationships are among the simplest relationships possible, and we’ve now seen that many phenomena in nature are linear (plus many more are at least approximately linear over some range). For these reasons, it’s a common practice to assume that two variables are linearly related if we don’t have any other information. This assumption can be quite useful when you try to make an order-of-magnitude estimate of some unknown quantity (see chapter 8). On the other hand, caution is advisable when drawing conclusions based on assumed linear dependence, as we saw in the case of dose-response relationships. Some phenomena turn out to be very nonlinear, and assuming linearity can lead to incorrect conclusions. Assumed linear dependence is an intelligent working hypothesis, but it needs to be checked by empirical tests. The simplicity of nature is remarkable, but not infinite.